Introduction to Chern-Simons theory

To define Chern-Simons theory, we need 3 ingredients:

- compact oriented 3-manifold M
- compact simple gauge group G
- P principal G bundle over M

In these lectures: $G = SU(2)$

$\rightarrow$ P is topologically trivial (SU(2) simply connected)

Denote by $\mathcal{A}_M$ the space of connections on P.
identify $\mathcal{A}_M$ with $\Omega^1(M, \mathfrak{g})$ Lie-algebra valued 1-forms
coordinate notation: $A_i^a \in \mathcal{A}_M$

i: tangent to M

a: Lie-algebra generator

<u>Definition</u> (gauge transformation):

$\mathcal{G} := \mathrm{Map}(M, G)$ space of smooth maps from M to G

$$g^* A = g^{-1} A g + g^{-1} dg, \quad A \in \mathcal{A}_M,\ g \in \mathcal{G}$$

infinitesimal gauge transformation: $A_i \longmapsto A_i - D_i \varepsilon$
with $D_i \varepsilon = \partial_i \varepsilon + [A_i, \varepsilon]$

Definition (curvature):

$$F_A = dA + A \wedge A \in \Omega^2(M, \mathfrak{g})$$

Definition (Chern-Simons functional):

For $A \in \mathcal{A}_M$ we put

$$CS(A) = \frac{1}{8\pi^2} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

Proposition 1:

A critical point of the Chern-Simons functional is a flat connection.

Proof:

Consider a one-parameter family of connections $A_t = A + ta$. Then

$$CS(A + ta) = CS(A) + \frac{t}{4\pi^2} \int_M \mathrm{Tr}(F_A \wedge a) + O(t^2)$$

(exercise)

$\Rightarrow CS$ is critical at $A \Leftrightarrow F_A = 0$

$\square$

Proposition 2:

Let M be a compact oriented 3-manifold with $\partial M \neq 0$. Then we have

$$CS(g^* A) = CS(A) + \frac{1}{8\pi^2} \int_{\partial M} \mathrm{Tr}(A \wedge dg\, g^{-1}) - \int_M g^* \sigma$$

where σ is the volume form of $SU(2)$:

$$g^*\sigma = \frac{1}{24\pi^2} \mathrm{Tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right)$$

(exercise)

Consider now <u>2 cases</u>:

a) $\partial M = \phi \Rightarrow CS(g^*A) = CS(A) - \int_M g^*\sigma$

The integrand $\int_M g^*\sigma$ is considered as the mapping degree of the map $g: M \to SU(2)$ and is an integer ($\pi_3(SU(2)) = \mathbb{Z}$)

$$\longrightarrow CS : \mathcal{A}_M / \mathcal{G} \longrightarrow \mathbb{R}/\mathbb{Z}$$

b) $\partial M \neq \phi$, set $\partial M = \Sigma$

<u>Quantization:</u>

- classical mechanics: trajectory of particles determined by path minimizing the action integral S
  in Chern-Simons theory: $S = CS(A)$
  $\longrightarrow$ stationary points of S : $F_A = 0$

- quantum mechanics:
  any path γ contributes with the probability $e^{\sqrt{-1}\,S(\gamma)/\hbar}$
  $\longrightarrow$ "Feynman's path integral":
  $$\int e^{\sqrt{-1}\,S(\gamma)/\hbar}\, d\mu(\gamma)$$

where $d\mu$ is the measure on the measure on the space of paths connecting the above two points.
For $\hbar \to 0$ surviving contributions are critical points of $S(\gamma)$.
in Chern-Simons theory:

a) $\partial M = \emptyset$

$$Z_k(M) = \int_{\mathcal{A}_M/\mathcal{G}} \exp\left(2\pi\sqrt{-1}\, k\, CS(A)\right) \mathcal{D}A$$

"Witten-invariant"

where $k \in \mathbb{Z}$ (follows from Prop. 2)

b) $\partial M = \Sigma$

take a G connection on Σ

consider $\mathcal{A}_{M,\alpha} := \{ A \in \mathcal{A}_M \mid A|_\Sigma = \alpha \}$

let $Z_k(M)_\alpha$ be the restriction of the pathintegral in a) to $\mathcal{A}_{M,\alpha}$

Will show: $Z_k(M)_\alpha$ is section of line bundle $\mathcal{L}$

$$\begin{array}{c} \mathcal{L} \\ \downarrow \\ \mathcal{M}_{G,\Sigma} \end{array}$$

(moduli space of flat G-bundles over Σ)

sections of $\mathcal{L}$ are elements of Hilbert space of a 2d QFT on Σ
"conformal field theory"

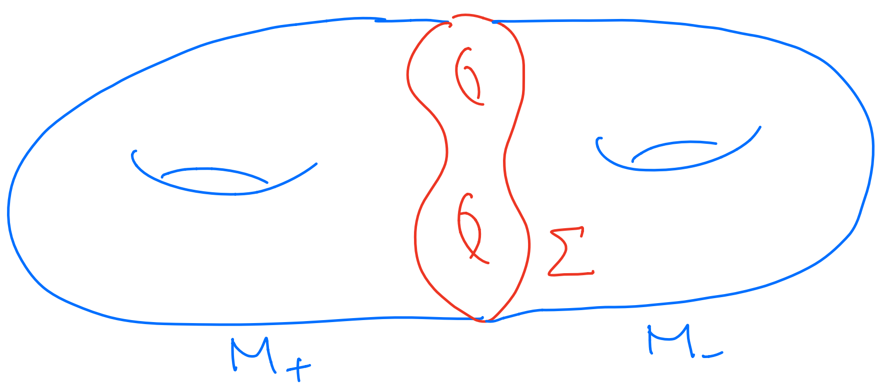
Next, consider the situation

$$M = M_+ \cup M_-$$

with $\partial M_+ = \Sigma$ and $\partial M_- = -\Sigma$

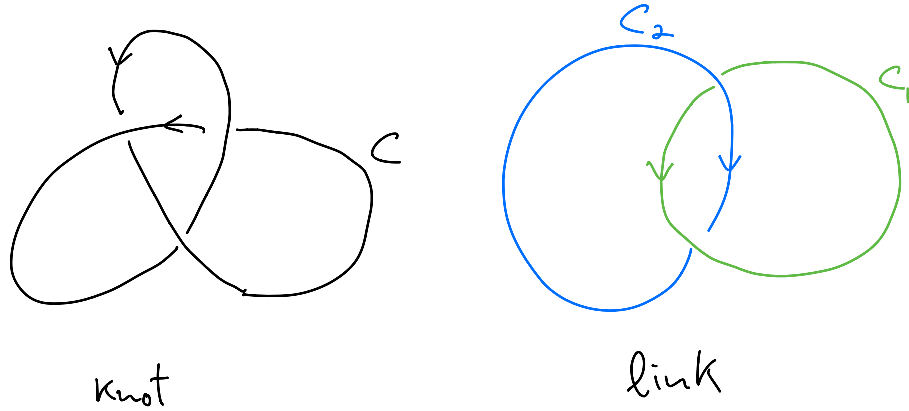
<u>goal:</u> $Z_k(M) = \langle Z_k(M_+), Z_k(M_-) \rangle$

by "integrating" over α



<u>Inclusion of Wilson lines:</u>

Let C_j, $1 \leq j \leq r$ be the components of a "link" L in M. Then each C_j is a "knot"



Definition (Wilson line operator):

Assign a representation R_j of the Lie Group G to each component C_j. Then

$$W_{C_j,R_j}(A) = \mathrm{Tr}_{R_j} P\exp\int_{C_j} A\,dx = \mathrm{Tr}_{R_j} \mathrm{Hol}_{C_j}(A)$$

where P denotes "path-ordering":

$$P\exp\int_{C_j} A\,dx = \sum_{n=0}^{\infty} \int_0^t \cdots \int_0^{t'_{n-1}} A(t'_1)\cdots A(t'_n)\,dt'_1 \cdots dt'_n$$

where $t' \in [0,t]$ being a parametrization of C_j

Definition (Witten's invariant with Wilson lines):

$$Z_k(M;C_1,\dots,C_r) = \int \exp\left(2\pi\sqrt{-1}\,k\,CS(A)\right)\prod_{j=1}^{r} W_{C_j,R_j}(A)\,\mathcal{D}A$$